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TRANSVERSE VIBRATION OF A RECTANGULAR PLATE WITH AN ECCENTRIC CIRCULAR INNER BOUNDARY

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Abstract—In this paper a method for solving vibration problems of a rectangular plate with an eccentric circular inner boundary is presented. The eigenvalue problem of the plate is solved by the use of the exact solution of the equation of motion which satisfies the inner boundary conditions. The boundary conditions along the outer edge are satisfied by means of the Fourier expansion method. Numerical calculations are carried out for various combinations of outer and inner boundary conditions, and the nondimensional natural frequencies are given for a number of cases.

1. INTRODUCTION

A rectangular plate having an inner circular boundary is widely used in many branches of engineering. The dynamic behavior of this type of plate therefore needs to be understood and many investigations on the vibration of square[1, 2], rectangular[3, 4] and irregular shaped[5] plates with holes have been reported. However the plate with the other edges and that with eccentric inner edges have not been thoroughly discussed although the solution in Ref. [4] imposes no symmetry restrictions. This paper deals with eigenvalue problems of a rectangular plate with an eccentric circular inner edge for various combinations of outer and inner boundary conditions. For this problem, general approximate methods such as the finite element, finite difference and point matching methods[4] are usually applied. These methods have many advantages for solving vibration problems of a plate with irregular boundaries. However in such methods a large size digital computer is required and, in general, there are many computational difficulties to obtain good results in cases of higher mode vibrations[6, 7]. Therefore it seems to be important to give a more straightforward method from which the results with reasonable accuracy being obtained easily by using a minicomputer. Recently the author gave a straightforward method to deal with vibration and dynamic response problems of membranes[8–11] and plates[11, 12] with arbitrary shape. In those studies the boundary of the plate was restricted to one made up of curves. In this paper this method is expanded into vibration problems of the rectangular plate with an eccentric circular inside edge. In the analysis the exact solution of equation of motion which satisfies the inner boundary conditions is utilized and the boundary conditions along the outer straight line boundary are satisfied directly by means of the Fourier expansion method. Numerical calculations are carried out for the plates with various edge conditions. To verify the present analysis, the results on the problem treated by the previous authors are also obtained as a special case of this problem.

2. ANALYSIS

A rectangular plate having an eccentric circular inner boundary is shown in Fig. 1, in which Cartesian coordinates x, y and polar coordinates r, θ are taken as in the figure. The equation of motion for a plate, in the polar coordinates r, θ is

$$D\nabla^2\nabla^2 w + \rho h \partial^2 w/\partial t^2 = 0 \tag{1}$$

where w is the transverse displacement, ∇^2 is the two-dimensional Laplacian operator, ρ is the mass density, h is the thickness, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, E is Young's modulus, ν is Poisson's ratio and t is the time. The displacement w is denoted by $w = W(r, \theta) \sin \omega t$ for free vibrations. Hence we have

$$W = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \epsilon_n [A_{jn} J_n(\alpha r) + B_{jn} Y_n(\alpha r) + C_{jn} I_n(\alpha r) + D_{jn} K_n(\alpha r)] \Phi_{jn} \tag{2}$$

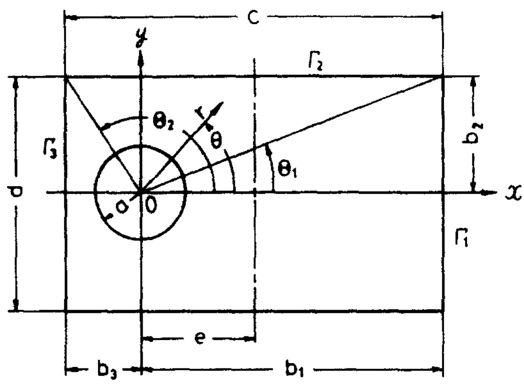


Fig. 1. Geometry of a rectangular plate with an eccentric circular inside edge.

where

$$\begin{aligned}&\epsilon_n = 1/2 \quad \text{for } n = 0, \qquad \text{and} \qquad \epsilon_n = 1 \quad \text{for } n \geqq 1\\&\Phi_{1n} = \cos n\theta, \quad \Phi_{2n} = \sin n\theta, \quad \alpha^4 = \rho h\omega^2/D\end{aligned} \tag{3}$$

ω is the circular frequency, A_{jn}, B_{jn}, C_{jn} and D_{jn} are constants of integration to be determined from the boundary conditions, $J_n(\alpha r)$ and $Y_n(\alpha r)$ are Bessel functions of first and second kinds of order n and $I_n(\alpha r)$ and $K_n(\alpha r)$ are modified Bessel functions. The boundary conditions along the inner edge are

$$\begin{aligned}&(w)_{r=a} = (\partial w/\partial r)_{r=a} = 0 && \text{for the clamped dege}\\&(M_r)_{r=a} = (V_r)_{r=a} = 0 && \text{for the free edge}\\&(w)_{r=a} = (M_r)_{r=a} = 0 && \text{for the simply supported edge}\end{aligned} \tag{4}$$

where M_r is the bending moment and V_r is the Kirchoff shear in the plate. Substituting eqn (2) into eqns (4) and eliminating the constants C_{jn} and D_{jn}, we have

$$\begin{aligned}W = \sum_{j=1}^{2} \sum_{n=0}^{\infty} \epsilon_n [A_{jn}\{J_n(\alpha r) + \gamma_{3n} I_n(\alpha r) + \gamma_{1n} K_n(\alpha r)\}\\+ B_{jn}\{Y_n(\alpha r) + \gamma_{4n} I_n(\alpha r) + \gamma_{2n} K_n(\alpha r)\}]\Phi_{jn}.\end{aligned} \tag{5}$$

The coefficients $\gamma_{1n} \sim \gamma_{4n}$ are given in the Appendix for various inner boundary conditions.

The boundary conditions along the outer edge are

$$\begin{aligned}&(w)_\Gamma = (\partial w/\partial z)_\Gamma = 0 && \text{for the clamped edge}\\&(w)_\Gamma = (M_z)_\Gamma = 0 && \text{for the simply supported edge}\\&(V_z)_\Gamma = (M_z)_\Gamma = 0 && \text{for the free edge}\end{aligned} \tag{6}$$

where $(w)_\Gamma$ is the displacement at the boundary Γ and z is the coordinate normal to the boundary. Since the displacement is expressed in terms of the coordinates r and θ, it is convenient to treat the outer boundary conditions when the derivatives in the equations of the bending slope, bending moment and shearing force which are expressed as functions of x and y are transformed in terms of the coordinates r and θ. If we consider two important cases of clamped and simply supported edges, the transformed expressions are

$$\begin{aligned}
\partial W/\partial x &= (\partial W/\partial r)\cos\theta - (\partial W/\partial\theta)(1/r)\sin\theta \\
\partial W/\partial y &= (\partial W/\partial r)\sin\theta + (\partial W/\partial\theta)(1/r)\cos\theta \\
M_x &= -D(\partial^2 W/\partial x^2 + \nu\partial^2 W/\partial y^2) \\
&= -D[(\partial^2 W/\partial r^2)(\cos^2\theta + \nu\sin^2\theta) - 2\{(\partial^2 W/\partial\theta\partial r)(1-\nu)/r\}\sin\theta\cos\theta \\
&\quad + (\partial W/\partial r)(\sin^2\theta + \nu\cos^2\theta)/r + 2\{(\partial W/\partial\theta)(1-\nu)/r^2\}\sin\theta\cos\theta \\
&\quad + (\partial^2 W/\partial\theta^2)(\sin^2\theta + \nu\cos^2\theta)/r^2] \\
M_y &= -D(\partial^2 W/\partial y^2 + \nu\partial^2 W/\partial x^2) \\
&= -D[(\partial^2 W/\partial r^2)(\nu\cos^2\theta + \sin^2\theta) + 2\{(\partial^2 W/\partial\theta\partial r)(1-\nu)/r\}\sin\theta\cos\theta \\
&\quad + (\partial W/\partial r)(\nu\sin^2\theta + \cos^2\theta)/r - 2\{(\partial W/\partial\theta)(1-\nu)/r^2\}\sin\theta\cos\theta \\
&\quad + (\partial^2 W/\partial\theta^2)(\nu\sin^2\theta + \cos^2\theta)/r^2].
\end{aligned} \tag{7}$$

Equations (7) are the exact transformed expressions of bending slopes and bending moments in the x and y directions. Substituting eqn (5) into eqns (7) yields

$$\begin{aligned}
\partial W/\partial x &= \alpha \sum_{j=1}^{2}\sum_{n=0}^{\infty} \epsilon_n[A_{jn}X_{jn}(r,\theta) + B_{jn}L_{jn}(r,\theta)] \\
\partial W/\partial y &= \alpha \sum_{j=1}^{2}\sum_{n=0}^{\infty} \epsilon_n[A_{jn}\bar{X}_{jn}(r,\theta) + B_{jn}\bar{L}_{jn}(r,\theta)] \\
M_x &= -D\alpha^2 \sum_{j=1}^{2}\sum_{n=0}^{\infty} \epsilon_n[A_{jn}G_{jn}(r,\theta) + B_{jn}H_{jn}(r,\theta)] \\
M_y &= -D\alpha^2 \sum_{j=1}^{2}\sum_{n=0}^{\infty} \epsilon_n[A_{jn}\bar{G}_{jn}(r,\theta) + B_{jn}\bar{H}_{jn}(r,\theta)]
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
X_{1n}(r,\theta) &= J'_n(\alpha r)\cos\theta\cos n\theta + (n/\alpha r)J_n(\alpha r)\sin\theta\sin n\theta + [I'_n(\alpha r)\cos\theta\cos n\theta \\
&\quad + (n/\alpha r)I_n(\alpha r)\sin\theta\sin n\theta]\gamma_{3n} + [K'_n(\alpha r)\cos\theta\cos n\theta + (n/\alpha r)K_n(\alpha r)\sin\theta \\
&\quad \times \sin n\theta]\gamma_{1n} \\
L_{1n}(r,\theta) &= Y'_n(\alpha r)\cos\theta\cos n\theta + (n/\alpha r)Y_n(\alpha r)\sin\theta\sin n\theta + [I'_n(\alpha r)\cos\theta\cos n\theta \\
&\quad + (n/\alpha r)I_n(\alpha r)\sin\theta\sin n\theta]\gamma_{4n} + [K'_n(\alpha r)\cos\theta\cos n\theta + (n/\alpha r)K_n(\alpha r) \\
&\quad \times \sin\theta\sin n\theta]\gamma_{2n} \\
\bar{X}_{1n}(r,\theta) &= J'_n(\alpha r)\sin\theta\cos n\theta - (n/\alpha r)J_n(\alpha r)\cos\theta\sin n\theta + [I'_n(\alpha r)\sin\theta\cos n\theta \\
&\quad - (n/\alpha r)I_n(\alpha r)\cos\theta\sin n\theta]\gamma_{3n} + [K'_n(\alpha r)\sin\theta\cos n\theta - (n/\alpha r)K_n(\alpha r) \\
&\quad \times \cos\theta\sin n\theta]\gamma_{1n} \\
\bar{L}_{1n}(r,\theta) &= Y'_n(\alpha r)\sin\theta\cos n\theta - (n/\alpha r)Y_n(\alpha r)\cos\theta\sin n\theta + [I'_n(\alpha r)\sin\theta\cos n\theta \\
&\quad - (n/\alpha r)I_n(\alpha r)\cos\theta\sin n\theta]\gamma_{4n} + [K'_n(\alpha r)\sin\theta\cos n\theta - (n/\alpha r)K_n(\alpha r)\cos\theta\sin n\theta]\gamma_{2n} \\
G_{1n}(r,\theta) &= g_{1n} + \gamma_{3n}g_{3n} + \gamma_{1n}g_{4n}, \quad H_{1n}(r,\theta) = g_{2n} + \gamma_{4n}g_{3n} + \gamma_{2n}g_{4n} \\
\bar{G}_{1n}(r,\theta) &= \bar{g}_{1n} + \gamma_{3n}\bar{g}_{3n} + \gamma_{1n}\bar{g}_{4n}, \quad \bar{H}_{1n}(r,\theta) = \bar{g}_{2n} + \gamma_{4n}\bar{g}_{3n} + \gamma_{2n}\bar{g}_{4n}
\end{aligned} \tag{9}$$

and where

$$J_n'(\alpha r) = J_{n-1}(\alpha r) - (n/\alpha r)J_n(\alpha r), \quad Y_n'(\alpha r) = Y_{n-1}(\alpha r) - (n/\alpha r)Y_n(\alpha r)$$

$$I_n'(\alpha r) = I_{n-1}(\alpha r) - (n/\alpha r)I_n(\alpha r), \quad K_n'(\alpha r) = -[K_{n-1}(\alpha r) + (n/\alpha r)K_n(\alpha r)]$$

$$g_{1n} = [(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 - (\cos^2\theta + \nu \sin^2\theta)\cos n\theta]J_n(\alpha r)$$
$$-[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]J_{n-1}(\alpha r)$$

$$g_{2n} = [(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 - (\cos^2\theta + \nu \sin^2\theta)\cos n\theta]Y_n(\alpha r)$$
$$-[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]Y_{n-1}(\alpha r)$$

$$g_{3n} = [(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 + (\cos^2\theta + \nu \sin^2\theta)\cos n\theta]I_n(\alpha r)$$
$$-[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]I_{n-1}(\alpha r)$$

$$g_{4n} = [(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 + (\cos^2\theta + \nu \sin^2\theta)\cos n\theta]K_n(\alpha r)$$
$$+[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]K_{n-1}(\alpha r)$$

$$\bar{g}_{1n} = [-(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 - (\nu\cos^2\theta + \sin^2\theta)\cos n\theta]J_n(\alpha r)$$
$$+[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]J_{n-1}(\alpha r)$$

$$\bar{g}_{2n} = [-(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 - (\nu\cos^2\theta + \sin^2\theta)\cos n\theta]Y_n(\alpha r)$$
$$+[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]Y_{n-1}(\alpha r)$$

$$\bar{g}_{3n} = [-(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 + (\nu\cos^2\theta + \sin^2\theta)\cos n\theta]I_n(\alpha r)$$
$$+[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]I_{n-1}(\alpha r)$$

$$\bar{g}_{4n} = [-(n^2+n)(1-\nu)(\cos 2\theta \cos n\theta - \sin 2\theta \sin n\theta)/\alpha^2 r^2 + (\nu\cos^2\theta + \sin^2\theta)\cos n\theta]K_n(\alpha r)$$
$$-[(1-\nu)(\cos 2\theta \cos n\theta - n \sin 2\theta \sin n\theta)/\alpha r]K_{n-1}(\alpha r). \quad (10)$$

The expressions for $j = 2$ have the same forms as eqns (10), but $\cos n\theta$ is replaced by $\sin n\theta$ and also $\sin n\theta$ is replaced by $-\cos n\theta$.

The boundary conditions along the whole range of the outer edge cannot be satisfied directly because the expressions for the conditions are not expressed as trigonometric series of the coordinate θ. To satisfy the outer boundary conditions, the Fourier expansion is performed to the expressions along the boundary lines. In the present case, since the boundary is consisted of four straight lines and has corners, the boundary is separated between the corners. Fourier coefficients are therefore obtained by the addition of these for the separated boundaries. When the plate is symmetric about the x-axis, the motion of it is separated into two types of symmetric and antisymmetric vibrations. In this case, if we take θ as an independent variable, the boundary conditions are expressed in the following Fourier series:

$$\left.\begin{aligned}\sum_m \sum_n \epsilon_n \epsilon_m (S_{nm}^{1j} A_{jn} + S_{nm}^{2j} B_{jn})\psi_{jm} = 0 \\ \sum_m \sum_n \epsilon_n \epsilon_m (S_{nm}^{3j} A_{jn} + S_{nm}^{4j} B_{jn})\psi_{jm} = 0\end{aligned}\right\} \quad \text{for } j = 1, 2 \quad (11)$$

where $\psi_{1m} = \cos m\theta$, $\psi_{2m} = \sin m\theta$, $\epsilon_m = 1/2$ for $m = 0$ and $\epsilon_m = 1$ for $m \geqq 1$. By using the symmetry, $S_{nm}^{1j} \sim S_{nm}^{4j}$ are obtained as

$$S_{nm}^{kj} = (2/\pi) \sum_{i=1}^{3} \int_{\Theta_{i-1}}^{\Theta_i} Z_{jn}^{ik}(\theta)\psi_{jm}\, d\theta \qquad \text{for } k = 1, 2, 3, 4 \quad (12)$$

where $\Theta_0 = 0$, $\Theta_3 = \pi$ and

$$\begin{aligned}
Z^{11}_{jn}(\theta) &= [J_n(\alpha R_i) + \gamma_{3n} I_n(\alpha R_i) + \gamma_{1n} K_n(\alpha R_i)]\Phi_{jn} \\
Z^{12}_{jn}(\theta) &= [Y_n(\alpha R_i) + \gamma_{4n} I_n(\alpha R_i) + \gamma_{2n} K_n(\alpha R_i)]\Phi_{jn} \\
Z^{13}_{jn}(\theta) &= X_{jn}(R_1, \theta), \quad Z^{23}_{jn}(\theta) = \bar{X}_{jn}(R_2, \theta), \quad Z^{33}_{jn}(\theta) = X_{jn}(R_3, \theta) \\
Z^{14}_{jn}(\theta) &= L_{jn}(R_1, \theta), \quad Z^{24}_{jn}(\theta) = \bar{L}_{jn}(R_2, \theta), \quad Z^{34}_{jn}(\theta) = L_{jn}(R_3, \theta)
\end{aligned} \tag{13}$$

for the clamped edge, and

$$\begin{aligned}
Z^{13}_{jn}(\theta) &= G_{jn}(R_1, \theta), \quad Z^{23}_{jn}(\theta) = \bar{G}_{jn}(R_2, \theta), \quad Z^{33}_{jn}(\theta) = G_{jn}(R_3, \theta) \\
Z^{14}_{jn}(\theta) &= H_{jn}(R_1, \theta), \quad Z^{24}_{jn}(\theta) = \bar{H}_{jn}(R_2, \theta), \quad Z^{34}_{jn}(\theta) = H_{jn}(R_3, \theta)
\end{aligned} \tag{14}$$

for the simply supported edge. $Z^{i1}_{jn}(\theta)$ and $Z^{i2}_{jn}(\theta)$ have the same forms as those shown in eqns (13) in the case of simply supported edges. R_i is the coordinate r at ith boundary which is expressed as a function of θ.

The frequency equation is obtained from eqns (11) for the symmetric mode with respect to the x-axis by taking $j = 1$. When the terms n and m are truncated to $N+1$, we have

$$\begin{vmatrix}
S^{11}_{00} & S^{11}_{10} \cdots S^{11}_{N0} & S^{21}_{00} & S^{21}_{10} \cdots S^{21}_{N0} \\
S^{11}_{01} & \ddots & S^{21}_{01} & \ddots \\
S^{11}_{0N} & \cdots S^{11}_{NN} & S^{21}_{0N} & \cdots S^{21}_{NN} \\
S^{31}_{00} & S^{31}_{10} \cdots S^{31}_{N0} & S^{41}_{00} & S^{41}_{10} \cdots S^{41}_{N0} \\
S^{31}_{01} & \ddots & S^{41}_{01} & \ddots \\
S^{31}_{0N} & \cdots S^{31}_{NN} & S^{41}_{0N} & \cdots S^{41}_{NN}
\end{vmatrix} = 0. \tag{15}$$

The frequency equation for the antisymmetric mode with respect to the x-axis can be obtained by taking $j = 2$, n, $m = 1,2,3,\ldots N$.

3. NUMERICAL RESULTS

We have the following relations (see Fig. 1)

$$\begin{aligned}
&\Theta_0 = 0, \quad \Theta_1 = \tan^{-1}(b_2/b_1), \quad \Theta_2 = \pi - \tan^{-1}(b_2/b_3), \quad \Theta_3 = \pi \\
&R_1 = b_1/\cos\theta, \quad R_2 = b_2/\sin\theta, \quad R_3 = -b_3/\cos\theta, \quad b_1 = e + c/2, \quad b_2 = d/2, \quad b_3 = c/2 - e.
\end{aligned} \tag{16}$$

Substituting eqns (16) into eqns (13) and (14), Z^{ik}_{jn} is expressed as a function of θ only. Then the integration in eqn (12) can be performed numerically. In this analysis the outer boundary conditions are satisfied by using an infinite number of Fourier series and hence the results obtained must be investigated with respect to both the convergence of the series and the errors of numerical calculation. Table 1 depicts the fundamental nondimensional natural frequency $\lambda(=\omega d^2\sqrt{(\rho h/D)})$ of a square plate with a central hole vs N. Since the convergence of the series is good, the results with sufficient accuracy can be obtained when up to seven terms are included in the numerical calculation. When the value of a/c reduces to zero, the solution becomes to that for the plate with a point hole and hence the fundamental frequency should be a little lower than in the case of the solid plates. However the discrepancy between the result for $a/c = 0$ and that of the solid plate is significantly small. To verify the present analysis, the result for small values of a/c is compared with that of a solid plate and in the case of the clamped plate, the result $\lambda = 35.984$ is obtained for $a/c = 0.025$, while that for the solid plate is $\lambda = 35.99$[13]. The present result in such a special case shows good agreement with Young's result. Figures 2 and 3 show the comparison between author's and other authors' results[3, 4] for

Table 1. Fundamental natural frequencies $\lambda(=\omega d^2\sqrt{(\rho h/D)})$ of a square plate with a centrol hole vs N for $\nu = 0.3$

	Clamped plates			Simply supported		
	a/d			a/d		
N	0.1	0.2	0.3	0.1	0.2	0.3
3	34.96	35.53	38.21			
4	36.12	36.76	39.67			
5	35.81	36.48	39.36	19.53	19.29	19.50
6	35.83	36.51	39.39	19.53	19.29	19.50
7	35.83	36.51	39.39	19.52	19.29	19.50

fundamental frequencies in the case of a square plate with a central hole. Since Takahashi, in using the Rayleigh–Ritz method, employed as his comparison functions a set of functions divided from beam deflection functions (i.e. Poisson's ratio $\nu = 0$), the shape of the curves given by the author and Hegarty and Ariman is different from that by Takahashi. Detailed discussions about it were given by Hegarty and Ariman [4] using the point matching method. It can be noted that the present results are in good agreement with those by Hegarty and Ariman, and discrepancies between both analyses are limited within 2.5%. It seems therefore the present analysis gives reliable results for vibration problems of a plate with a circular inner boundary.

Tables 2 and 3 show the fundamental natural frequencies $\lambda(=\omega d^2\sqrt{(\rho h/D)})$ of the plate with a square outer boundary of length $c = d$ and an eccentric circular inner boundary of eccentricity e for various combinations of the outer and the inner boundary conditions. The abbreviation C denotes the clamped edge, S the simply supported edge and F the free edge. Tables 4 and 5 show the frequencies $\lambda(=\omega d^2\sqrt{(\rho h/D)})$ of a rectangular plate with an eccentric circular inner boundary. The effects of the eccentricity on the natural frequencies are small in the case of the plate with a hole, while when the inner edge is a clamped or a supported boundary, the frequency varies significantly, and those effects cannot be neglected. Figures 4 and 5 show the frequencies for first and second modes vs the eccentricity e/c in the case of a clamped square plate with a hole and the plate with a clamped circular inner boundary. It can

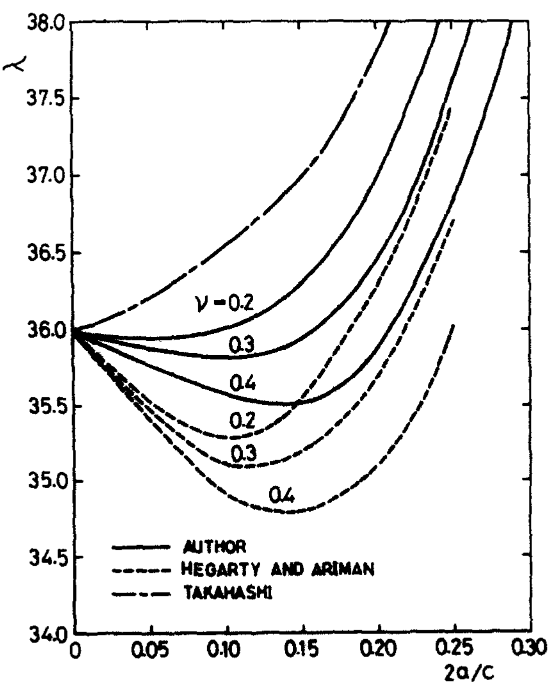


Fig. 2. Comparison between author's and other authors' results for fundamental natural frequencies λ of a clamped square plate with a central hole.

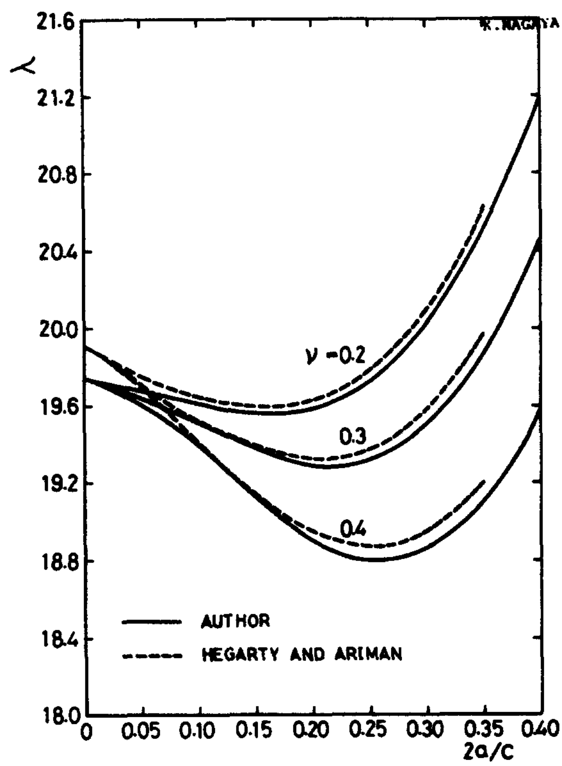


Fig. 3. Comparison between author's and other authors' results for fundamental natural frequencies λ for a simply supported square plate with a central hole.

Table 2. Fundamental natural frequencies λ of a clamped square plate with an eccentric circular inside edge for $\nu = 0.3$

		e/c						
	$2a/c$	0	0.05	0.10	0.15	0.20	0.25	0.30
C ○ C	0.1	90.63	81.81	72.11	64.02	57.42	52.08	47.77
	0.2	109.8	97.41	84.62	73.86	65.17	58.20	52.61
	0.3	133.3	117.0	100.6	86.42	74.95	65.83	58.54
C ○ S	0.1	77.06	71.20	63.64	57.23	51.98	47.73	44.39
	0.2	87.77	79.83	70.30	62.33	55.92	50.77	46.67
	0.3	105.2	93.42	80.73	70.30	61.98	55.42	50.20
C ○ F	0.1	35.83	35.85	35.88	35.93	35.93	35.88	35.77
	0.2	36.51	36.51	36.46	36.38	36.24	36.01	35.79
	0.3	39.39	39.14	38.49	37.63	36.73	35.91	35.62

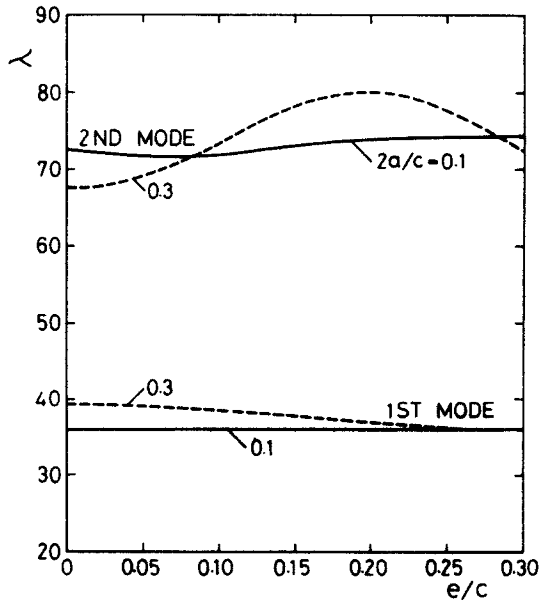


Fig. 4. Frequencies λ vs eccentricities for a clamped square plate having an eccentric circular hole with $\nu = 0.3$.

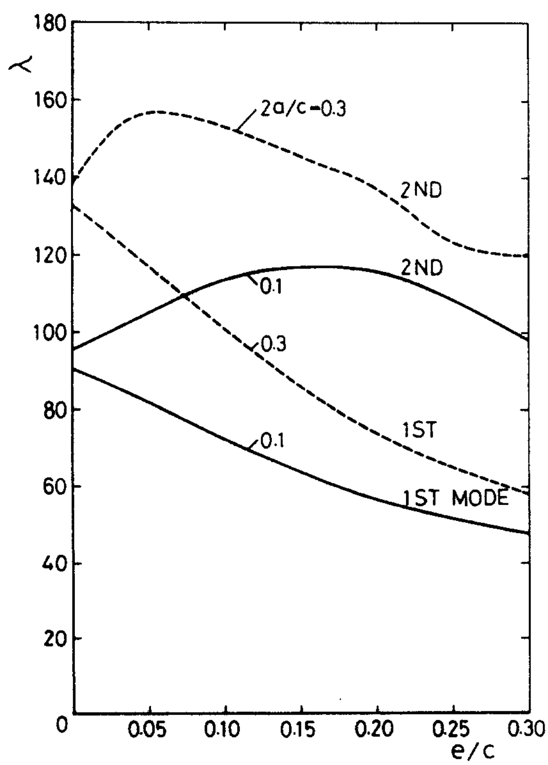


Fig. 5. Frequencies λ vs eccentricities for a clamped square plate having an eccentric circular clamped inside edge.

Table 3. Fundamental natural frequencies λ of a simply supported square plate with an eccentric circular inside edges for $\nu = 0.3$

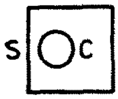
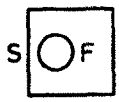
		e/c						
	2a/c	0	0.05	0.10	0.15	0.20	0.25	0.30
	0.1	60.42	54.83	48.48	43.11	38.61	34.92	31.86
	0.2	73.27	65.39	56.98	49.89	44.11	39.39	35.53
	0.3	89.64	78.80	67.79	58.48	50.92	44.83	39.95
	0.1	50.75	47.17	42.33	38.08	34.48	31.45	28.95
	0.2	56.78	52.17	46.23	41.14	36.92	33.42	30.55
	0.3	67.53	60.64	52.73	46.14	40.83	36.52	32.98
	0.1	19.52	19.53	19.55	19.56	19.60	19.62	19.63
	0.2	19.29	19.29	19.32	19.35	19.40	19.44	19.45
	0.3	19.50	19.49	19.47	19.42	19.38	19.32	19.25

Table 4. Fundamental natural frequencies λ of a clamped rectancular plate with an eccentric circular hole for $\nu = 0.3$

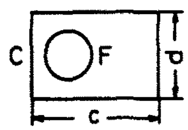
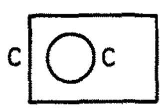
			e/c						
	c/d	2a/c	0	0.05	0.10	0.15	0.20	0.25	0.30
	1.5	0.1	26.98	27.05	27.12	27.26	27.27	26.98	26.42
		0.2	28.30	28.30	28.30	28.23	27.95	27.26	26.56
		0.3	32.05	31.77	30.87	29.96	28.92	27.74	26.84
	2.0	0.1	24.58	24.72	24.87	25.13	25.50	25.88	24.62
		0.25	28.38	28.28	28.03	27.75	27.45	26.90	25.18
		0.3	30.48	29.55	28.88	28.38	27.93	27.43	25.88

Table 5. Fundamental natural frequencies λ of a clamped rectangular plate with an eccentric clamped circular inside edge

			e/c						
	c/d	2a/c	0	0.05	0.10	0.15	0.20	0.25	0.30
	2.0	0.1	38.78	34.38	32.03	30.65	30.25	30.65	29.35
		0.2	42.85	37.15	33.75	31.73	31.20	31.43	29.60
		0.3	50.60	42.28	37.05	33.98	32.73	32.10	31.30

be observed that the frequency for the first mode decreases with the eccentricity, while that for the second mode increases. From these discussions, it is concluded that the effects of the eccentricity on the frequencies increase as the rigidity of the inner edge increases, and in general, those effects cannot be neglected. The results given in the tables and figures denote the frequencies of symmetric modes with respect to the x-axis. The frequencies of antisymmetric modes can also be obtained in the same way.

4. CONCLUSIONS

In this paper a method for solving vibration problems of a plate having a rectangular outside and an eccentric circular inside edge has been presented. The frequency equation of the plate has been obtained and numerical calculations have been carried out for various combinations of outer and inner boundary conditions. It is concluded that the effects of the eccentricity of the inner edge on the natural frequencies increase as the rigidity of the inner boundary becomes large and in general, these effects cannot be neglected. The convergence of the series was good and the results with sufficient accuracy were obtained easily using a minicomputer. Therefore this method has some advantages compared with the other general approximate methods.

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APPENDIX

The coefficients $\gamma_{1n} \sim \gamma_{4n}$ in eqn (5) are given by

$$\gamma_{1n} = (j_{1n}i_{2n} - j_{2n}i_{1n})/F, \quad \gamma_{2n} = (y_{1n}i_{2n} - y_{2n}i_{1n})/F$$
$$\gamma_{3n} = -(j_{1n}k_{2n} - j_{2n}k_{1n})/F, \quad \gamma_{4n} = -(y_{1n}k_{2n} - y_{2n}k_{1n})/F$$
$$F = i_{1n}k_{2n} - k_{1n}i_{2n} \tag{17}$$

where

$$j_{1n} = J_n(\alpha r), \quad j_{2n} = J_{n-1}(\alpha a) - (n/\alpha a)J_n(\alpha a), \quad y_{1n} = Y_n(\alpha a)$$
$$y_{2n} = Y_{n-1}(\alpha a) - (n/\alpha a)Y_n(\alpha a), \quad i_{1n} = I_n(\alpha a), \quad i_{2n} = I_{n-1}(\alpha a) - (n/\alpha a)I_n(\alpha a)$$
$$k_{1n} = K_n(\alpha a), \quad k_{2n} = -K_{n-1}(\alpha a) - (n/\alpha a)K_n(\alpha a) \tag{18}$$

for the clamped edge, and

$$j_{1n} = m_{1n} = [n(n+1)(1-\nu)/\alpha^2a^2 - 1]J_n(\alpha a) - [(1-\nu)/\alpha a]J_{n-1}(\alpha a)$$
$$y_{1n} = m_{2n} = [n(n+1)(1-\nu)/\alpha^2a^2 - 1]Y_n(\alpha a) - [(1-\nu)/\alpha a]Y_{n-1}(\alpha a)$$
$$i_{1n} = m_{3n} = [n(n+1)(1-\nu)/\alpha^2a^2 + 1]I_n(\alpha a) - [(1-\nu)/\alpha a]I_{n-1}(\alpha a)$$
$$k_{1n} = m_{4n} = [n(n+1)(1-\nu)/\alpha^2a^2 + 1]K_n(\alpha a) + [(1-\nu)/\alpha a]K_{n-1}(\alpha a)$$
$$j_{2n} = [n^2(n+1)(1-\nu)/\alpha^3a^3 + n/\alpha a]J_n(\alpha a) - [n^2(1-\nu)/\alpha^2a^2 + 1]J_{n-1}(\alpha a)$$
$$y_{2n} = [n^2(n+1)(1-\nu)/\alpha^3a^3 + n/\alpha a]Y_n(\alpha a) - [n^2(1-\nu)/\alpha^2a^2 + 1]Y_{n-1}(\alpha a)$$
$$i_{2n} = [n^2(n+1)(1-\nu)/\alpha^3a^3 - n/\alpha a]I_n(\alpha a) - [n^2(1-\nu)/\alpha^2a^2 - 1]I_{n-1}(\alpha a)$$
$$k_{2n} = [n^2(n+1)(1-\nu)/\alpha^3a^3 - n/\alpha a]K_n(\alpha a) + [n^2(1-\nu)/\alpha^2a^2 - 1]K_{n-1}(\alpha a) \tag{19}$$

for the free edge and

$$j_{1n} = J_n(\alpha a), \quad j_{2n} = m_{1n}, \quad y_{1n} = Y_n(\alpha a), \quad y_{2n} = m_{2n}$$
$$i_{1n} = I_n(\alpha a), \quad i_{2n} = m_{3n}, \quad k_{1n} = K_n(\alpha a), \quad k_{2n} = m_{4n} \tag{20}$$

for the simply supported edge.